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Exact Solutions for Stimulated Emissions by External Sources

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The exact solutions for transition amplitudes are derived for *stimulated* emissions by external sources. More precisely, we obtain the exact expressions for transition amplitudes for the emission of an arbitrary number of particles by the sources when some particles are already present, in the process, *prior* to the "switching on" of the external sources. The solutions are given for an arbitrary number of particles with arbitrary configurations (of momenta, spin, etc.) and for particles of spin-0, spin-1/2, massive and massless (photons) spin-1 particles, and massless (gravitons) spin-2 particles. Applications are given as illustrations to the process $\phi \to$ anything, and, in quantum electrodynamics, to the process $\gamma \to e^+e^- +$ any photons, in the *presence* of external sources, where a (virtual) photon decays into the pair e^+e^-.

1. INTRODUCTION

Almost all the statistical properties of multiparticle emission by external sources (cf., Manoukian, 1984, for a recent study; see also Paul, 1982) may be obtained from the known expressions (cf., Schwinger, 1970, 1969) of the transition amplitudes, in the presence of the sources, from the vacuum to the multiparticle state. The latter expressions may be obtained from the vacuum-to-vacuum transition amplitude $\langle 0_+|0_-\rangle$ followed by a systematic use of unitarity. The treatment of the problem for *stimulated* particle emission by the sources is, however, more involved but is in the same spirit and a systematic study of the problem is certainly lacking in the literature. By stimulated particle emission one is referring to transition amplitudes for particle production by the sources when some particles are already present, in the process, *prior* to the "switching on" of the external sources. Expressions are known to exist in the literature (cf., Schwinger, 1970) only

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for very special cases, such as for weak sources, but general expressions which cover all cases, such as for strong sources and with arbitrary number of particles of different configurations (of momenta, spin, etc.) in the initial *and* final states, are, to our knowledge, not available. The purpose of this paper is to obtain such *exact* expressions. Our main results are summarized in equations (16), (26), and (54). The analysis is applied to particles of spin 0, massive and massless (photons) spin-1 particles, massless (gravitons) spin-2 particles, and for particles of spin 1/2. Applications are carried out in Section 3 as illustrations to the process $\phi \to$ anything, and, in quantum electrodynamics, to the process $\gamma \to e^+e^-$ + any photons, where a (virtual) photon decays into the pair e^+e^-.

2. STIMULATED EMISSION

2.1. Spin-0 Particles

The vacuum-to-vacuum transition amplitude for charge-0, spin-0 particles interacting with an external source $K(x)$ is given by the well-known expression

$$\langle 0_+|0_-\rangle^K = \exp\frac{i}{2}\int (dx)(dx')K(x)\Delta_+(x-x')K(x') \equiv \exp\frac{i}{2}K\Delta_+K \quad (1)$$

where

$$\Delta_+(x-x') = \int \frac{(dp)}{(2\pi)^4}\frac{e^{ip(x-x')}}{(p^2+m^2-i\varepsilon)}, \qquad \varepsilon \to +0 \quad (2)$$

$$\Delta_+(x-x') = i\int d\omega_p\, e^{ip(x-x')},\ x^0 > x'^0, \qquad d\omega_{\mathbf{p}} = \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0} \quad (3)$$

$p^0 = +(\mathbf{p}^2+m^2)^{1/2}$. We also introduce the Fourier transform:

$$K(p) = \int (dx)\, e^{-ixp}K(x) \quad (4)$$

For the subsequent analysis, it is convenient to introduce a discretization notation (Schwinger, 1969, 1970; Manoukian, 1984) for the momentum variable by setting in the process:

$$K_{\mathbf{p}} = (d\omega_{\mathbf{p}})^{1/2}K(p) \quad (5)$$

The vacuum persistence probability may be then written as

$$|\langle 0_+|0_-\rangle^K|^2 = \exp -\sum_{\mathbf{p}} |K_{\mathbf{p}}|^2 \quad (6)$$

Let $\{\mathbf{p}_1, \mathbf{p}_2, \ldots\} = S$ denote the set of all possible momenta in a convenient discrete-momentum notation, If $n_{\mathbf{p}_1}, n_{\mathbf{p}_2}, \ldots$ denote the number of particles with momenta $\mathbf{p}_1, \mathbf{p}_2, \ldots$, such that $n_{\mathbf{p}_1}+n_{\mathbf{p}_2}+\cdots=n$ denotes the total number of particles, then (Schwinger, 1970; Manoukian, 1984):

$$\langle n; n_{\mathbf{p}_1}, n_{\mathbf{p}_2}, \ldots |0_-\rangle^K = \langle 0_+|0_-\rangle^K \frac{(iK_{\mathbf{p}_1})^{n_{\mathbf{p}1}}}{(n_{\mathbf{p}_1}!)^{1/2}} \frac{(iK_{\mathbf{p}_2})^{n_{\mathbf{p}2}}}{(n_{\mathbf{p}_2}!)^{1/2}} \cdots \tag{7}$$

$$\langle 0_+|n; n_{\mathbf{p}_1}, n_{\mathbf{p}_2}, \ldots\rangle^K = \langle 0_+|0_-\rangle^K \frac{(iK^*_{\mathbf{p}_1})^{n_{\mathbf{p}1}}}{(n_{\mathbf{p}_1}!)^{1/2}} \frac{(iK^*_{\mathbf{p}_2})^{n_{\mathbf{p}2}}}{(n_{\mathbf{p}_2}!)^{1/2}} \cdots \tag{8}$$

To obtain the transition amplitudes for stimulated emission we proceed as follows. We write (Schwinger, 1970) $K = K_1 + K_2 + K_3$, where the source K_2 is switched on after the source K_1 is switched off, and the source K_3 is switched on after the source K_2 is switched off. We may then write

$$\langle 0_+|0_-\rangle^K = \langle 0_+|0_-\rangle^{K_3}\langle 0_+|0_-\rangle^{K_2}\langle 0_+|0_-\rangle^{K_1} \exp iK_3^* iK_2 \exp iK_3^* iK_1 \exp iK_2^* iK_1 \tag{9}$$

where

$$iK_3^* iK_2 \equiv \sum_{\mathbf{p}} iK^*_{3\mathbf{p}} iK_{2\mathbf{p}} \tag{10}$$

and $K_{\mathbf{p}}$ is defined in (5), (4). Upon expanding the last three exponentials on the right-hand side of (9), we obtain

$$\begin{aligned}\langle 0_+|0_-\rangle^K = \sum_{***} & \frac{(iK^*_{3\mathbf{p}_1})^{n_{\mathbf{p}1}}}{(n_{\mathbf{p}_1}!)^{1/2}} \frac{(iK^*_{3\mathbf{p}_2})^{n_{\mathbf{p}2}}}{(n_{\mathbf{p}_2}!)^{1/2}} \cdots \frac{(iK^*_{3\mathbf{p}_1})^{m_{\mathbf{p}1}}}{(m_{\mathbf{p}_1}!)^{1/2}} \frac{(iK^*_{3\mathbf{p}_2})^{m_{\mathbf{p}2}}}{(m_{\mathbf{p}_2}!)^{1/2}} \\ & \times \cdots \langle 0_+|0_-\rangle^{K_3} \frac{(iK_{2\mathbf{p}_1})^{n_{\mathbf{p}1}}}{(n_{\mathbf{p}_1}!)^{1/2}} \frac{(iK_{2\mathbf{p}_2})^{n_{\mathbf{p}2}}}{(n_{\mathbf{p}_2}!)^{1/2}} \cdots \frac{(iK^*_{2\mathbf{p}_1})^{l_{\mathbf{p}1}}}{(l_{\mathbf{p}_1}!)^{1/2}} \\ & \times \frac{(iK^*_{2\mathbf{p}_2})^{l_{\mathbf{p}2}}}{(l_{\mathbf{p}_2}!)^{1/2}} \cdots \langle 0_+|0_-\rangle^{K_2} \frac{(iK_{1\mathbf{p}_1})^{m_{\mathbf{p}1}}}{(m_{\mathbf{p}_1}!)^{1/2}} \frac{(iK_{1\mathbf{p}_2})^{m_{\mathbf{p}2}}}{(m_{\mathbf{p}_2}!)^{1/2}} \cdots \\ & \times \frac{(iK_{1\mathbf{p}_1})^{l_{\mathbf{p}1}}}{(l_{\mathbf{p}_1}!)^{1/2}} \frac{(iK_{1\mathbf{p}_2})^{l_{\mathbf{p}2}}}{(l_{\mathbf{p}_2}!)^{1/2}} \cdots \langle 0_+|0_-\rangle^{K_1}\end{aligned} \tag{11}$$

where $\sum_{***}$ stands for a summation over all nonnegative integers $n_{\mathbf{p}_1}$, $n_{\mathbf{p}_2}$, $\ldots$, $m_{\mathbf{p}_1}$, $m_{\mathbf{p}_2}$, $\ldots$, $l_{\mathbf{p}_1}$, $l_{\mathbf{p}_2}$, $\ldots$, such that

$$n_{\mathbf{p}_1}+n_{\mathbf{p}_2}+\cdots=n,\ m_{\mathbf{p}_1}+m_{\mathbf{p}_2}+\cdots=m,\ l_{\mathbf{p}_1}+l_{\mathbf{p}_2}+\cdots=l \tag{12}$$

and over all $m, n, l = 0, 1, 2, \ldots$.

We may also rewrite $\langle 0_+|0_-\rangle^K$ in terms of a unitarity sum (Schwinger, 1970, Manoukian, 1984) as

$$\begin{aligned}\langle 0_+|0_-\rangle^K = \sum_{**} & \langle 0_+|N; N_{\mathbf{p}_1}, N_{\mathbf{p}_2}, \ldots\rangle^{K_3} \\ & \times \langle N; N_{\mathbf{p}_1}, N_{\mathbf{p}_2}, \ldots|M; M_{\mathbf{p}_1}, M_{\mathbf{p}_2}, \ldots\rangle^{K_2}\langle M; M_{\mathbf{p}_1}, M_{\mathbf{p}_2}, \ldots|0_-\rangle^{K_1}\end{aligned} \tag{13}$$

where $\sum_{**}$ stands for a summation over all nonnegative integers $N_{\mathbf{p}_1}$, $N_{\mathbf{p}_2}, \ldots, M_{\mathbf{p}_1}, M_{\mathbf{p}_2}, \ldots$ such that

$$N_{\mathbf{p}_1}+N_{\mathbf{p}_2}+\cdots=N, \qquad M_{\mathbf{p}_1}+M_{\mathbf{p}_2}+\cdots=M \tag{14}$$

and over all N, $M=0,1,2,\ldots$. The amplitude $\langle N; N_{\mathbf{p}_1}, N_{\mathbf{p}_2}, \ldots | M; M_{\mathbf{p}_1}, M_{\mathbf{p}_2}, \ldots\rangle^{K_2}$ represents the transition amplitude for the creation of N particles, $N_{\mathbf{p}_1}$ of which have momenta $\mathbf{p}_1$, $N_{\mathbf{p}_2}$ of which have momenta $\mathbf{p}_2, \ldots$, when there are already M particles, $M_{\mathbf{p}_1}$ of which have momenta $\mathbf{p}_1$, $M_{\mathbf{p}_2}$ of which have momenta $\mathbf{p}_2, \ldots$, prior to the switching on of the source K_2. This is exactly the object we are seeking and represents the stimulated emission of particles.

Upon systematic use of the expressions for the amplitudes in equations (7), (8), setting

$$n_{\mathbf{p}_i}+m_{\mathbf{p}_i}=N_{\mathbf{p}_i}, \qquad m_{\mathbf{p}_i}+l_{\mathbf{p}_i}=M_{\mathbf{p}_i} \tag{15}$$

$i=1,2,\ldots$, and comparing equations (11) and (13) we arrive to the following expression:

$$\begin{aligned}
&\langle N; N_{\mathbf{p}_1}, N_{\mathbf{p}_2}, \ldots | M; M_{\mathbf{p}_1}, M_{\mathbf{p}_2}, \ldots\rangle^{K} \\
&\quad=(N_{\mathbf{p}_1}!N_{\mathbf{p}_2}!\ldots M_{\mathbf{p}_1}!M_{\mathbf{p}_2}!\ldots)^{1/2}\sum{}^{*}\frac{(iK_{\mathbf{p}_1})^{N_{\mathbf{p}1}-m_{\mathbf{p}1}}}{(N_{\mathbf{p}_1}-m_{\mathbf{p}_1})!} \\
&\quad\times\frac{(iK_{\mathbf{p}_2})^{N_{\mathbf{p}2}-m_{\mathbf{p}2}}}{(N_{\mathbf{p}_2}-m_{\mathbf{p}_2})!}\cdots\frac{\langle 0_+|0_-\rangle^K}{m_{\mathbf{p}_1}!m_{\mathbf{p}_2}!\cdots}\frac{(iK^*_{\mathbf{p}_1})^{M_{\mathbf{p}1}-m_{\mathbf{p}1}}}{(M_{\mathbf{p}_1}-m_{\mathbf{p}_1})!}\frac{(iK^*_{\mathbf{p}_2})^{M_{\mathbf{p}2}-m_{\mathbf{p}2}}}{(M_{\mathbf{p}_2}-m_{\mathbf{p}_2})!}\cdots
\end{aligned} \tag{16}$$

for a general source K, and where $\sum^*$ stands for a summation over all nonnegative integers $m_{\mathbf{p}_1}, m_{\mathbf{p}_2}, \ldots$, *such that* $0\le m_{\mathbf{p}_i}\le\min(N_{\mathbf{p}_i}, M_{\mathbf{p}_i})$, $i=1,2,\ldots$, and where the equalities in equation (14) should be noted. The expression in (16) is exact. Applications of this formula will be given in Section 3.

2.2. Massive Spin-1 Particles

The vacuum-to-vacuum transition amplitude is given by (cf. Schwinger, 1969), in the presence of an external source $J^\mu(x)$:

$$\langle 0_+|0_-\rangle^J=\exp\frac{i}{2}\int(dx)(dx')J^\mu(x)\left(g_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2}\right)\Delta_+(x-x')J^\nu(x') \tag{17}$$

$$g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}=\sum_{\lambda=1}^{3}e_\mu(p,\lambda)e_\nu(p,\lambda)^* \tag{18}$$

$$p^\mu e_\mu(p,\lambda)=0, \qquad \lambda=1,2,3 \tag{19}$$

$$e^\mu(p,\lambda)^*e_\mu(p,\lambda')=\delta_{\lambda\lambda'}, \qquad \lambda,\lambda'=1,2,3 \tag{20}$$

where $e^{\mu}(p, \lambda)$ is the polarization vector. Upon setting

$$J_{\mathbf{p}\lambda} = (d\omega_{\mathbf{p}})^{1/2} e_{\mu}(p, \lambda)^* J^{\mu}(p), \qquad d\omega_{\mathbf{p}} = \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2p^0}, \qquad p^0 = (\mathbf{p}^2 + m^2)^{1/2} \tag{21}$$

$$J^{\mu}(p) = \int (dx)\, e^{-ipx} J^{\mu}(x) \tag{22}$$

we have in a convenient discrete-momentum notation:

$$|\langle 0_+|0_-\rangle^J|^2 = \exp\left(-\sum_{\mathbf{p},\lambda} |J_{\mathbf{p}_2}|^2\right), \qquad \lambda = 1, 2, 3 \tag{23}$$

Upon setting $r = (\mathbf{p}, \lambda)$, $r \in \{r_1 = (\mathbf{p}_1, 1),\ r_2 = (\mathbf{p}_1, 2),\ r_3 = (\mathbf{p}_1, 3),\ r_4 = (\mathbf{p}_2, 1), \ldots\}$ one may also write, for example, directly from (7), (8)

$$\langle n; n_{r_1}, n_{r_2}, \ldots|0_-\rangle^J = \langle 0_+|0_-\rangle^J \frac{(iJ_{r_1})^{n_{r_1}}}{(n_{r_1}!)^{1/2}} \frac{(iJ_{r_2})^{n_{r_2}}}{(n_{r_2}!)^{1/2}} \cdots \tag{24}$$

$$\langle 0_+|n; n_{r_1}, n_{r_2}, \ldots\rangle^J = \langle 0_+|0_-\rangle^J \frac{(iJ^*_{r_1})^{n_{r_1}}}{(n_{r_1}!)^{1/2}} \frac{(iJ^*_{r_2})^{n_{r_2}}}{(n_{r_2}!)^{1/2}} \cdots \tag{25}$$

Finally from (16) we may then infer that

$$\begin{aligned} &\langle N; N_{r_1}, N_{r_2}, \ldots|M; M_{r_1}, M_{r_2}, \ldots\rangle^J \\ &= (N_{r_1}! N_{r_2}! \cdots M_{r_1}! M_{r_2}! \cdots)^{1/2} \sum{}^* \frac{(iJ_{r_1})^{N_{r_1}-m_{r_1}}}{(N_{r_1}-m_{r_1})!} \frac{(iJ_{r_2})^{N_{r_2}-m_{r_2}}}{(N_{r_2}-m_{r_2})!} \\ &\times \cdots \frac{\langle 0_+|0_-\rangle^J}{m_{r_1}! m_{r_2}! \cdots} \frac{(iJ^*_{r_1})^{M_{r_1}-m_{r_1}}}{(M_{r_1}-m_{r_1})!} \frac{(iJ^*_{r_2})^{M_{r_2}-m_{r_2}}}{(M_{r_2}-m_{r_2})!} \cdots \end{aligned} \tag{26}$$

where $\sum^*$ stands for a summation over all nonnegative integers m_{r_i}, $i = 1, 2, \ldots$, such that $0 \le m_{r_i} \le \min(N_{r_i}, M_{r_i})$. Also note that $N_{r_1} + N_{r_2} + \cdots = N$, $M_{r_1} + M_{r_2} + \cdots = M$, and $J_r \equiv J_{\mathbf{p}\lambda}2$, with $J_{\mathbf{p}\lambda}$ defined in (21).

2.3. Photons

The vacuum-to-vacuum transition amplitude, in the presence of a conserved external current $J^{\mu}(x)$: $\partial_{\mu} J^{\mu}(x) = 0$, $p^{\mu} J_{\mu}(p) = 0$, is given by (cf., Schwinger, 1970):

$$\langle 0_+|0_-\rangle^J = \exp \frac{i}{2} \int (dx)(dx') J^{\mu}(x) D_+(x - x') J_{\mu}(x') \tag{27}$$

$$D_{\mu\nu}(x - x') = \int \frac{(dp)}{(2\pi)^4} \frac{e^{ip(x-x')}}{p^2 - i\varepsilon}, \qquad \varepsilon \to +0 \tag{28}$$

$$g^{\mu\nu} = \frac{p^{\mu}\bar{p}^{\nu} + \bar{p}^{\mu}p^{\nu}}{(p\bar{p})} + \sum_{\lambda=1}^{2} e^{\mu}(p, \lambda) e^{\nu}(p, \lambda)^* \tag{29}$$

$p^\mu = (p^0, \mathbf{p})$, $\bar{p}^\mu = (p^0, -\mathbf{p})$, where $e^\mu(p, \lambda)$ is the polarization vector, $\lambda = 1, 2$. Upon setting

$$J_{\mathbf{p}\lambda} = (d\omega_{\mathbf{p}})^{1/2} e_\mu(p, \lambda)^* J^\mu(p), \qquad d\omega_{\mathbf{p}} = \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2|\mathbf{p}|} \tag{30}$$

We may write

$$|\langle 0_+|0_-\rangle^J|^2 = \exp\left(-\sum_{\mathbf{p},\lambda} |J_{\mathbf{p}\lambda}|^2 \right), \qquad \lambda = 1, 2 \tag{31}$$

Equations (24), (25), and (26) also hold true with now $\lambda = 1, 2$, $m \equiv 0$.

2.4. Massless Spin-2 Particles: Gravitons

The vacuum-to-vacuum transition amplitude, in the presence of a symmetric $T_{\mu\nu}(x) = T_{\nu\mu}(x)$ and conserved $\partial_\mu T^{\mu\nu}(x) = 0$, $p_\mu T^{\mu\nu}(p) = 0$, $p_\mu T^{\mu\nu}(-p) = 0$, external source $T_{\mu\nu}(x)$ is given by (cf., Schwinger, 1970):

$$\langle 0_+|0_-\rangle^T = \exp \frac{i}{2} \int (dx)(dx') T^{\mu\nu}(x)[g_{\mu\lambda}g_{\nu\sigma} - \tfrac{1}{2}g_{\mu\nu}g_{\lambda\sigma}] D_+(x-x') T^{\lambda\sigma}(x') \tag{32}$$

We set

$$T_{\mathbf{p}\lambda} = (d\omega_{\mathbf{p}})^{1/2} e_{\mu\nu}(p, \lambda) T^{\mu\nu}(p), \qquad d\omega_{\mathbf{p}} = \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2|\mathbf{p}|} \tag{33}$$

where $e_{\mu\nu}(p, \lambda)$, $\lambda = 1, 2$ are (real) polarization tensors. We may effectively replace $[g_{\mu\lambda}g_{\nu\sigma} - \frac{1}{2}g_{\mu\nu}g_{\lambda\sigma}]$ as

$$[g_{\mu\lambda}g_{\nu\sigma} - \tfrac{1}{2}g_{\mu\nu}g_{\lambda\sigma}] \to \sum_{\lambda'=1}^{2} e_{\mu\nu}(p, \lambda') e_{\lambda\sigma}(p, \lambda')^* \tag{34}$$

One then obtains

$$|\langle 0_+|0_-\rangle|^2 = \exp\left(-\sum_{\mathbf{p},\lambda} |T_{\mathbf{p}\lambda}|^2 \right), \qquad \lambda = 1, 2 \tag{35}$$

and equations (24), (25), and (26) are true with J_r formally replaced by T_r, $r = (\mathbf{p}, \lambda)$, where now $\lambda = 1, 2$, $m \equiv 0$.

2.4. Spin-1/2 Particles

The vacuum-to-vacuum transition amplitude, in the presence of external (anticommuting) sources $\eta(x)$, $\bar{\eta}(x)$, may be written as

$$\langle 0_+|0_-\rangle^\eta = \exp i \int (dx)(dx') \bar{\eta}(x) S_+(x-x') \eta(x') \tag{36}$$

$$S_+(x-x') = \int \frac{(dp)}{(2\pi)^4} \frac{e^{ip(x-x')}(-\gamma p + m)}{p^2 + m^2 - i\varepsilon}, \qquad \varepsilon \to +0 \tag{37}$$

$(\gamma^0)^\dagger = \gamma^0$, $(\gamma_k)^\dagger = -\gamma_k$. It is readily checked that $\bar{\eta}(p)(-\gamma p+m)\eta(p)$ is real. Using the eigenfunction expansion

$$\frac{(-\gamma p+m)}{2m} = \sum_\sigma u(p,\sigma)\bar{u}(p,\sigma) \tag{38}$$

we may write $\bar{\eta}(p)(-\gamma p+m)\eta(p) = \sum_\sigma 2m\bar{\eta}(p)u(p,\sigma)\bar{u}(p,\sigma)\eta(p)$, and the latter is positive. Hence we have

$$|\langle 0_+|0_-\rangle|^2 = \exp\left\{-2\sum_\sigma \int d\omega_{\mathbf{p}}\, 2m[\bar{\eta}(p)u(p,\sigma)\bar{u}(p,\sigma)\eta(p)]\right\} \tag{39}$$

We also note in a standard notation:

$$\sum_\sigma v(p,\sigma)\bar{v}(p,\sigma) = -\frac{(\gamma p+m)}{2m} \tag{40}$$

$$\sum_\sigma [u(p,\sigma)\bar{u}(p,\sigma) - v(p,\sigma)\bar{v}(p,\sigma)] = 1 \tag{41}$$

$$\bar{u}(p,\sigma)u(p,\sigma') = \delta_{\sigma\sigma'} \tag{42}$$

$$\bar{v}(p,\sigma)v(p,\sigma') = -\delta_{\sigma\sigma'} \tag{43}$$

We set

$$(2m\, d\omega_{\mathbf{p}})^{1/2}\bar{\eta}(p)u(p,\sigma) = \eta^*_{\mathbf{p}\sigma-} \tag{44}$$

$$(2m\, d\omega_{\mathbf{p}})^{1/2}\bar{u}(p,\sigma)\eta(p) = \eta_{\mathbf{p}\sigma-} \tag{45}$$

$$(2m\, d\omega_{\mathbf{p}})^{1/2}\bar{v}(p,\sigma)\eta(-p) = \eta^*_{\mathbf{p}\sigma+} \tag{46}$$

$$(2m\, d\omega_{\mathbf{p}})^{1/2}\bar{\eta}(-p)v(p,\sigma) = \eta_{\mathbf{p}\sigma+} \tag{47}$$

where the signature $-$ corresponds to a particle, and $+$ to the anti-particle. We introduce the convenient notation $r \equiv (\mathbf{p}, \sigma, \varepsilon)$, $\sigma = 1, 2$, $\varepsilon = \pm$.

We then have

$$\langle n; n_{r_1}, n_{r_2}, \ldots|0_-\rangle^\eta = \frac{(i\eta_{r_1})^{n_{r_1}}}{(n_{r_1}!)^{1/2}}\frac{(i\eta_{r_2})^{n_{r_2}}}{(n_{r_2}!)^{1/2}}\cdots\langle 0_+|0_-\rangle^\eta \tag{48}$$

$$\langle 0_+|n; n_{r_1}, n_{r_2}, \ldots\rangle^\eta = \langle 0_+|0_-\rangle^\eta \cdots \frac{(i\eta^*_{r_2})^{n_{r_2}}}{(n_{r_2}!)^{1/2}}\frac{(i\eta^*_{r_1})^{n_{r_1}}}{(n_{r_1}!)^{1/2}} \tag{49}$$

where one should note the opposite ordering (Schwinger, 1970) in (48) and (49), which is a consequence of the anticommuting nature of the external sources. We note that $n_{r_i} = 0$ or 1 because $(\eta_r)^2 = 0$, and hence $n_r! = 1$ always.

From equations (48), (49) one should note that

$$\begin{aligned}(i\eta_{r_j})&\langle n-1; n_{r_1}, \ldots, n_{r_{j-1}}, 0, n_{r_{j+1}}, \ldots|0_-\rangle^\eta \\ &= (-1)^{n_{r_1}+\cdots+n_{r_{j-1}}}\langle n; n_{r_1}, \ldots, n_{r_{j-1}}, 1_{r_j}, n_{r_{j+1}}, \ldots|0_-\rangle^\eta\end{aligned} \tag{50}$$

$$\begin{aligned}\langle 0_+|&n-1; n_{r_1}, \ldots, n_{r_{j-1}}, 0, n_{r_{j+1}}, \ldots\rangle^\eta(i\eta^*_{r_j}) \\ &= (-1)^{n_{r_1}+\cdots+n_{r_{j-1}}}\langle 0_+|n; n_{r_1}, \ldots, n_{r_{j-1}}, 1_{r_j}, n_{r_{j+1}}, \ldots\rangle^\eta\end{aligned} \tag{51}$$

To obtain the transition amplitudes for stimulated emissions we write $\eta = \eta_1 + \eta_2 + \eta_3$, where η_2 is switched on after η_1 is switched off, and η_3 is switched on after η_2 is switched off. Upon using the expansion property ($n_{r_i} = 0$ or 1),

$$\begin{aligned}\exp \sum_r i\eta_{3r}^* i\eta_{2r} &= \sum_n \sum_{n_{r_1}+n_{r_2}+\cdots=n}' (i\eta_{3r_1}^*)^{n_{r_1}}(i\eta_{3r_2}^*)^{n_{r_2}} \cdots (i\eta_{2r_2})^{n_{r_2}}(i\eta_{2r_1})^{n_{r_1}} \\ &= \sum_n \sum_{n_{r_1}+n_{r_2}+\cdots=n}' (i\eta_{2r_1})^{n_{r_1}}(i\eta_{2r_2})^{n_{r_2}} \cdots (i\eta_{3r_2}^*)^{n_{r_2}}(i\eta_{3r_1}^*)^{n_{r_1}} \end{aligned} \tag{52}$$

we obtain, as in Eq. (11), the expression

$$\begin{aligned}\langle 0_+|0_-\rangle^\eta = \sum_{(***)} &\cdots (i\eta_{3r_2}^*)^{n_{r_2}}(i\eta_{3r_1}^*)^{n_{r_1}}(i\eta_{2r_1})^{n_{r_1}}(i\eta_{2r_2})^{n_{r_2}} \cdots \langle 0_+|0_-\rangle^{\eta_3} \\ &\times (i\eta_{3r_1}^*)^{m_{r_1}}(i\eta_{3r_2}^*)^{m_{r_2}} \cdots (i\eta_{2r_2})^{m_{r_2}}(i\eta_{2r_1})^{m_{r_1}}\langle 0_+|0_-\rangle^{\eta_2} \\ &\times \cdots (i\eta_{2r_2}^*)^{l_{r_2}}(i\eta_{2r_1}^*)^{l_{r_1}}(i\eta_{1r_1})^{l_{r_1}}(i\eta_{1r_2})^{l_{r_2}} \cdots \langle 0_+|0_-\rangle^{\eta_1}\end{aligned} \tag{53}$$

where one should be careful with the ordering, and $\sum_{(***)}$ stands for a summation over n_{r_i}, m_{r_i}, $l_{r_i} = 0, 1$; $i = 1, 2, \ldots$, such that $n_{r_1} + n_{r_2} + \cdots = n$, $m_{r_1} + m_{r_2} + \cdots = m$, $l_{r_1} + l_{r_2} + \cdots = l$ and $n, l, m = 0, 1, 2, \ldots$.

By a systematic use of equations (48)-(51), we obtain from (53) by using a unitarity expansion as in equation (13) the exact expression:

$$\begin{aligned}&\langle N; N_{r_1}, N_{r_2}, \ldots | M; M_{r_1}, M_{r_2}, \ldots \rangle^\eta \\ &\quad = \sum{}^* (-1)^{m_{r_1}(N+M)}(-1)^{m_{r_2}(N+M+N_{r_1}+M_{r_1})}(-1)^{m_{r_3}(N+M+N_{r_1}+N_{r_2}+M_{r_1}+M_{r_2})} \\ &\quad\quad \times \cdots (i\eta_{r_1})^{N_{r_1}-m_{r_1}}(i\eta_{r_2})^{N_{r_2}-m_{r_2}} \cdots \langle 0_+|0_-\rangle^\eta \\ &\quad\quad \cdots (i\eta_{r_2}^*)^{M_{r_2}-m_{r_2}}(i\eta_{r_1}^*)^{M_{r_1}-m_{r_1}}\end{aligned} \tag{54}$$

where $\sum^*$ stands for a summation over $m_{r_i} = 0$, 1 *such that* $0 \le m_{r_i} \le \min[N_{r_i}, M_{r_i}]$, $i = 1, 2, \ldots$. The presence of the phase factors in the summand in equation (54) should be noted.

3. EXAMPLES

Before considering applications to stimulated emission by the external sources, we quickly point out some of the details of the very general method developed earlier (Manoukian, 1984) in determining transition *probabilities* for multiparticle production by external sources.

Consider the transition amplitude in (7) in the presence of the real scalar source $K(x)$. The transition probability is then

$$|\langle n; n_{\mathbf{p}_1}, n_{\mathbf{p}_2}, \ldots |0_-\rangle^K|^2 = |\langle 0_+|0_-\rangle^K|^2 \frac{(|K_{\mathbf{p}_1}|^2)^{n_{\mathbf{p}_1}}}{n_{\mathbf{p}_1}!} \frac{(|K_{\mathbf{p}_2}|^2)^{n_{\mathbf{p}_2}}}{n_{\mathbf{p}_2}!} \cdots \tag{55}$$

where $|\langle 0_+|0_-\rangle|^2$ is given in (6). Let $\Delta_1 = \{\mathbf{Q}_{11}, \mathbf{Q}_{12}, \ldots\}, \ldots, \Delta_k = \{\mathbf{Q}_{k1}, \mathbf{Q}_{k2}, \ldots\}$ be some given disjoint subsets of the set of all momenta $S = \{\mathbf{p}_1, \mathbf{p}_2, \ldots\}$. Let $n_{\mathbf{Q}_{ij}}$ denote the number of particles with momenta $\mathbf{Q}_{ij}$. Let n_{Δ_i} denote the number of particles with momenta in the set Δ_i. Then we may use the following identity:

$$\sum_{n_{\mathbf{Q}_{11}}+n_{\mathbf{Q}_{12}}+\cdots=n_{\Delta_1}} \frac{(|K_{\mathbf{Q}_{11}}|^2)^{n_{\mathbf{Q}11}}}{n_{\mathbf{Q}_{11}}!} \frac{(|K_{\mathbf{Q}_{12}}|^2)^{n_{\mathbf{Q}12}}}{n_{\mathbf{Q}_{12}}!} \cdots$$

$$= \frac{[|K_{\mathbf{Q}_{11}}|^2 + |K_{\mathbf{Q}_{12}}|^2 + \cdots]^{n_{\Delta_1}}}{n_{\Delta_1}!} = \frac{\left(\sum_{\mathbf{Q}\in\Delta_1} |K_{\mathbf{Q}}|^2\right)^{n_{\Delta_1}}}{n_{\Delta_1}!}$$

$$\equiv \left(\int_{\Delta_1} d\omega_{\mathbf{Q}} \, |K(Q)|^2\right)^{n_{\Delta_1}} \Big/ n_{\Delta_1}! \tag{56}$$

Hence from (55), with $n = n_{\Delta_1} + \cdots + n_{\Delta_k}$, the transition probability that the source K emits n particles, n_{Δ_1} of them having momenta in Δ_1, n_{Δ_2} of them having momenta in $\Delta_2, \ldots$, is given by

$$\frac{\left(\int_{\Delta_1} d\omega_{\mathbf{Q}} \, |K(Q)|^2\right)^{n_{\Delta_1}}}{n_{\Delta_1}!} \cdots \frac{\left(\int_{\Delta_k} d\omega_{\mathbf{Q}} \, |K(Q)|^2\right)^{n_{\Delta_k}}}{n_{\Delta_k}!} \exp\left(-\int d\omega_{\mathbf{Q}} \, |K(Q)|^2\right) \tag{57}$$

We now consider some stimulated emission processes. We study the process $\phi \to$ any ϕ, in the presence of the external source K, where ϕ represents a charge-0, spin-0 particle. Suppose the initial momentum of ϕ is $\mathbf{p}_1$. In equation (16), this corresponds to $M = 1$, $M_{\mathbf{p}_1} = 1$, $M_{\mathbf{p}_2} = 0, \ldots$. Since $0 \leq m_{\mathbf{p}_i} \leq \min[N_{\mathbf{p}_i}, M_{\mathbf{p}_i}]$, this also means that $m_{\mathbf{p}_2} = 0$, $m_{\mathbf{p}_3} = 0, \ldots$, and $0 \leq m_{\mathbf{p}_1} \leq \min[N_{\mathbf{p}_1}, 1]$. That is

$$\langle N; N_{\mathbf{p}_1}, N_{\mathbf{p}_2}, \ldots | 1_{\mathbf{p}_1}\rangle^K = \frac{(iK_{\mathbf{p}_2})^{N_{\mathbf{p}2}}}{(N_{\mathbf{p}_2}!)^{1/2}} \frac{(iK_{\mathbf{p}_3})^{N_{\mathbf{p}3}}}{(N_{\mathbf{p}_3}!)^{1/2}} \cdots \langle 0_+|0_-\rangle^K iK^*_{\mathbf{p}_1} \tag{58}$$

if $N_{\mathbf{p}_1} = 0$, and for $N_{\mathbf{p}_1} \geq 1$,

$$\langle N; N_{\mathbf{p}_1}, N_{\mathbf{p}_2}, \ldots | 1_{\mathbf{p}_1}\rangle^K = (N_{\mathbf{p}_1}! N_{\mathbf{p}_2}! \cdots)^{1/2}$$

$$\times \left\{ \frac{(iK_{\mathbf{p}_1})^{N_{\mathbf{p}1}}}{N_{\mathbf{p}_1}!} \frac{(iK_{\mathbf{p}_2})^{N_{\mathbf{p}2}}}{N_{\mathbf{p}_2}!} \cdots \langle 0_+|0_-\rangle^K iK^*_{\mathbf{p}_1} + \frac{(iK_{\mathbf{p}1})^{N_{\mathbf{p}1}-1}}{(N_{\mathbf{p}_1}-1)!} \frac{(iK_{\mathbf{p}_2})^{N_{\mathbf{p}2}}}{N_{\mathbf{p}_2}!} \cdots \langle 0_+|0_-\rangle^K \right\} \tag{59}$$

Clearly, the second term in the curly brackets in equation (59) corresponds to a disconnected process where the initial particle just passes through the process undetected by the source. Hence in all cases, we may write for the connected process, where the initial particle is detected by the source before multiparticle production occurs is given by

$$\langle N; N_{\mathbf{p}_1}, N_{\mathbf{p}_2}, \ldots | 1_{\mathbf{p}_1}\rangle_c^K = \frac{(iK_{\mathbf{p}_1})^{N_{\mathbf{p}_1}}}{(N_{\mathbf{p}_1}!)^{1/2}} \frac{(iK_{\mathbf{p}_2})^{N_{\mathbf{p}_2}}}{(N_{\mathbf{p}_2}!)^{1/2}} \cdots \langle 0_+|0_-\rangle^K (iK^*_{\mathbf{p}_1}) \quad (60)$$

As an interesting application suppose that the net energy-momentum release is Q, then we may write for the transition *probability* for the process $\phi \to$ any ϕ, with masses $m = 0$:

$$(2\pi)^4 \sum_{N=0}^{\infty} \sum_{N_{\mathbf{p}_1}+N_{\mathbf{p}_2}+\cdots=N} \frac{(|K_{\mathbf{p}_1}|^2)^{N_{\mathbf{p}_1}}}{N_{\mathbf{p}_1}!} \frac{(|K_{\mathbf{p}_2}|^2)^{N_{\mathbf{p}_2}}}{N_{\mathbf{p}_2}!} \cdots |\langle 0_+|0_-\rangle^K|^2 |K_{\mathbf{p}_1}|^2$$
$$\times \delta(N_{\mathbf{p}_1} p_1 + N_{\mathbf{p}_2} p_2 + \cdots - p_1 - Q) \quad (61)$$

where $p_i = (|\mathbf{p}_i|, \mathbf{p}_i)$. The δ function may be more conveniently written as

$$\delta(N_{\mathbf{p}_1} p_1 + N_{\mathbf{p}_2} p_2 + \cdots - p_1 - Q)$$
$$= \frac{1}{(2\pi)^4} \int (dz) \exp i[N_{\mathbf{p}_1} p_1 + N_{\mathbf{p}_2} p_2 + \cdots - p_1 - Q]z \quad (62)$$

Hence upon using the identity in (56), we may rewrite (61) as

$$\int (dz) \sum_{N=0}^{\infty} \frac{[|K_{\mathbf{p}_1}|^2 e^{ip_1 z} + |K_{\mathbf{p}_2}|^2 e^{ip_2 z} + \cdots]^N}{N!} |K_{\mathbf{p}_1}|^2 \exp\left(-\sum_{\mathbf{p}} |K_{\mathbf{p}}|^2\right)$$
$$\times \exp[-i(p_1 + Q)z] \quad (63)$$

Therefore, if the tip of the initial particle momentum $\mathbf{p}_1 \equiv \mathbf{p}$ lies in the range $d^3\mathbf{p}$, we have from (63)

$$\frac{d^3\mathbf{p}}{(2\pi)^3} \frac{|K(p)|^2}{2p^0} \int (dz)\, e^{-i(p+Q)z} \exp\left[-\int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} |K(p)|^2 (1 - e^{ipz})\right] \quad (64)$$

where we have used the fact that

$$[|K_{\mathbf{p}_1}|^2 e^{ip_1 z} + |K_{\mathbf{p}_2}|^2 e^{ip_2 z} + \cdots] = \sum_{\mathbf{p}} |K_{\mathbf{p}}|^2 e^{ipz} \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0} |K(p)|^2 e^{ipz} \quad (65)$$

and the relation in equation (6).

We now consider a more involved application dealing with quantum electrodynamics in the presence of an external current $J^\mu(x)$. We study the process $\gamma \to e^+e^- +$ any photons, to lowest order in the charge, where a (virtual) photon decays into the pair e^+e^-. To this end the vacuum-to-vacuum transition amplitude may be effectively written as

$$\langle 0_+|0_-\rangle^{J,e} = \exp\frac{i}{2}\int (dx)(dx')J^\mu(x)D_{+(x-x')}^{(2)}J_\mu(x') \tag{66}$$

where

$$D_+^{(2)}(x-x') = \int\frac{(dp)}{(2\pi)^4}\int_0^\infty d\sigma^2\frac{\rho^{(2)}(\sigma^2)}{(p^2+\sigma^2-i\varepsilon)}, \qquad \varepsilon\to+0 \tag{67}$$

and we retain only the lowest contribution $\rho^{(2)}(\sigma^2)$, in e, to the spectral function. Also note that any multicurrent contribution in the exponential in $\langle 0_+|0_-\rangle^{J,e}$ necessarily involve terms of higher orders in e.

Upon setting effectively,

$$d\omega_{\mathbf{p}\sigma} = \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2(\mathbf{p}^2+\sigma^2)^{1/2}} \tag{68}$$

We may infer from equations (26), (64), and (66), that the transition probability P for the process is given by

$$P = \alpha\left(\frac{\partial}{\partial\alpha}P_\alpha\right)_{\alpha=0} \tag{69}$$

where $\alpha = e^2/4\pi$, and effectively,

$$\begin{aligned} P_\alpha = {} & \frac{d^2\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|}|e_\mu(p,\lambda)^*J^\mu(p)|^2\int (dz)\, e^{-i(p+Q)z} \\ & \times \exp\left[-\int\frac{d^3\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|}J^\mu(p)^*J_\mu(p)\right] \\ & \times\exp\left[\int\frac{d^3\mathbf{p}}{(2\pi)^3}\int_0^\infty\frac{d\sigma^2\rho^{(2)}(\sigma^2)}{2(\mathbf{p}^2+\sigma^2)^{1/2}}J^\mu(p)^*J_\mu(p)\right. \\ & \left.\times\exp\{i[\mathbf{p}\cdot\mathbf{z}-(\mathbf{p}^2+\sigma^2)^{1/2}z^0]\}\right] \end{aligned} \tag{70}$$

Using the well-known fact that

$$\left.\frac{\partial}{\partial\alpha}\rho(\sigma^2)\right|_{\alpha=0} = \left.\frac{\partial}{\partial\alpha}\rho^{(2)}(\sigma^2)\right|_{\alpha=0} = \frac{1}{3\pi\sigma^2}\left(1+\frac{2m^2}{\sigma^2}\right)\left(1-\frac{4m^2}{\sigma^2}\right)^{1/2} \tag{71}$$

$\sigma^2\geq 4m^2$, where m is the mass of the electron (positron), and if it has momentum p_1 (p_2), then $\sigma^2 = -(p_1+p_2)^2 = 2m^2-2(\mathbf{p}_1\cdot\mathbf{p}_2-p_1^0p_2^0)\geq 4m^2$. It is worth noting that $(\mathbf{p}_1+\mathbf{p}_2)^2+\sigma^2=(p_1^0+p_2^0)^2$.

All told the transition probability P for the process may be written

$$P=\left(\frac{\alpha}{3\pi}\right)\frac{d^3\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|}|e_\mu(p,\lambda)^* J^\mu(p)|^2\int(dz)\,e^{-i(p+Q)z}$$
$$\times\exp\left\{-\int\frac{d^3\mathbf{p}'}{(2\pi)^3 2|\mathbf{p}'|}[J^\mu(p')^* J_\mu(p')][1-e^{i(\mathbf{p}'\cdot\mathbf{z}-|\mathbf{p}'|z^0)}]\right\}$$
$$\times\int\frac{d^3\mathbf{q}}{(2\pi)^3}\int_{4m^2}^{\infty}\frac{d\sigma^2}{\sigma^2}\frac{1}{2(\mathbf{q}^2+\sigma^2)^{1/2}}\left(1+\frac{2m^2}{\sigma^2}\right)\left(1-\frac{4m^2}{\sigma^2}\right)^{1/2}$$
$$\times[J^\mu(q)^* J_\mu(q)]\exp i[\mathbf{q}\cdot\mathbf{z}-(\mathbf{q}^2+\sigma^2)^{1/2}z^0] \tag{72}$$

where the initial photon has a polarization λ, and a momentum $\mathbf{p}$ with tip in the range $d^3\mathbf{p}$. The momenta and the polarization of the final products are not measured. We note that in the last integral $q^0=(\mathbf{q}^2+\sigma^2)^{1/2}$.

Other applications are similarly carried out.

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